Problem 2.8

Use your result in Prob. 2.7 to find the field inside and outside a solid sphere of radius R that carries a uniform volume charge density ρ. Express your answers in terms of the total charge of the sphere, q. Draw a graph of $|\mathbf{E}|$ as a function of the distance from the center.

Solution

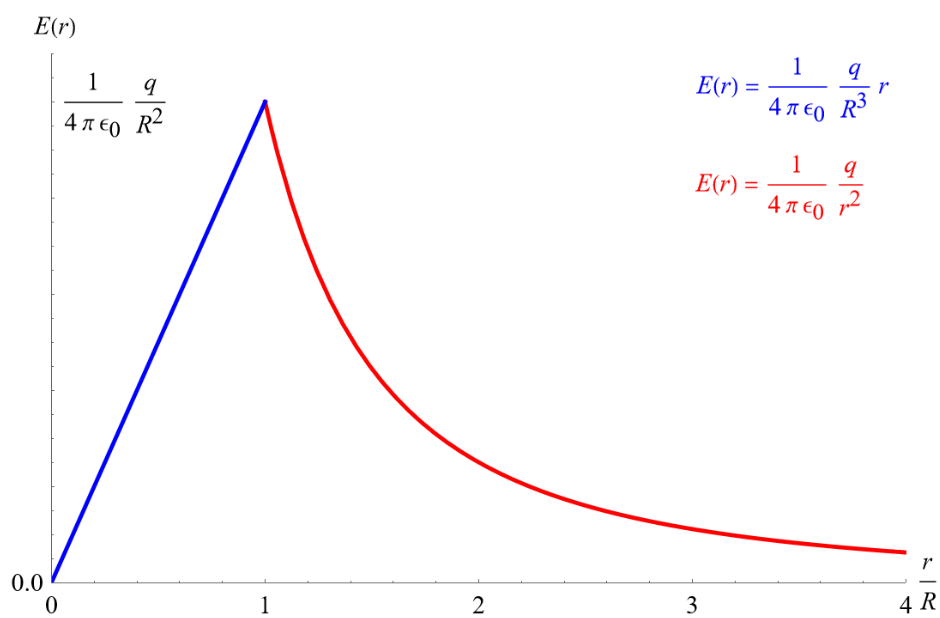
According to the result of Problem 2.7, the electric field outside a charged sphere is the same as if all charge were concentrated at its center. The electric field at radius r in a solid ball is the superposition of fields from every sphere with radius smaller than r; spheres with radius larger than r contribute nothing.

$$\mathbf{E}(r) = \frac{1}{4\pi\epsilon_0}\frac{Q_{\text{enclosed}}}{r^2}\hat{\mathbf{r}}, \quad \text{where } Q_{\text{enclosed}} = \begin{cases} \rho V(r) = \frac{q}{\frac{4}{3}\pi R^3}\left(\frac{4}{3}\pi r^3\right) = \frac{q}{R^3}r^3 & \text{if } r < R \\ \rho V(R) = \frac{q}{\frac{4}{3}\pi R^3}\left(\frac{4}{3}\pi R^3\right) = q & \text{if } r > R \end{cases}$$

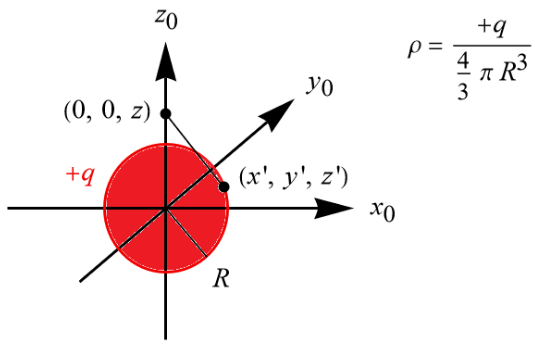
Therefore, the electric field at radius r is

$$\boxed{\mathbf{E} = \begin{cases} \dfrac{1}{4\pi\epsilon_0}\dfrac{q}{R^3}r\hat{\mathbf{r}} & \text{if } r < R \\ \\ \dfrac{1}{4\pi\epsilon_0}\dfrac{q}{r^2}\hat{\mathbf{r}} & \text{if } r > R \end{cases}}.$$

Below is a plot of the electric field magnitude $|\mathbf{E}| = E(r)$ versus r/R.



Calculating the electric field integral should give the same answer. Start by drawing a schematic for some point on the solid ball.



The formula for the electric field from a continuous distribution of charge in a volume is

$$\mathbf{E} = \frac{1}{4\pi\epsilon_0}\int \frac{\rho(\mathbf{r}')}{\mathscr{r}^2}\hat{\mathscr{r}}\,d\tau' = \frac{1}{4\pi\epsilon_0}\int \frac{\rho(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|^2}\left(\frac{\mathbf{r}-\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|}\right)d\tau'$$
$$= \frac{1}{4\pi\epsilon_0}\int \frac{\rho(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|^3}(\mathbf{r}-\mathbf{r}')\,d\tau',$$

where the integral is taken over the volume where the charge exists. Note that $\mathbf{r}$ is the position vector to where we want to know the electric field, $\mathbf{r}'$ is the position vector to the point we chose in the volume, and $\mathscr{r} = |\mathbf{r}-\mathbf{r}'|$ is the distance from the point we chose on the volume to where we want to know the electric field.

$$\mathbf{E} = \frac{1}{4\pi\epsilon_0}\iiint\limits_{x_0^2+y_0^2+z_0^2\le R^2} \frac{\rho}{\left[\sqrt{(0-x')^2+(0-y')^2+(z-z')^2}\right]^3}(\langle 0,0,z\rangle - \langle x',y',z'\rangle)\,dV'$$

The surface is spherical, so the appropriate parameterization is done with spherical coordinates (r_0, ϕ_0, θ_0), where θ_0 is the angle from the polar axis.

$$\mathbf{r}' = r'\langle \cos\phi' \sin\theta', \sin\phi' \sin\theta', \cos\theta' \rangle, \quad 0 \le r' \le R,\ 0 \le \phi' \le 2\pi,\ 0 \le \theta' \le \pi$$

Consequently, the electric field at $\mathbf{r} = \langle 0, 0, z \rangle$ is

$$\begin{aligned}
\mathbf{E} &= \frac{\rho}{4\pi\epsilon_0} \int_0^\pi \int_0^{2\pi} \int_0^R \frac{1}{\left[\sqrt{(0 - r'\cos\phi'\sin\theta')^2 + (0 - r'\sin\phi'\sin\theta')^2 + (z - r'\cos\theta')^2}\right]^3} (\langle 0,0,z\rangle - r'\langle \cos\phi'\sin\theta', \sin\phi'\sin\theta', \cos\theta'\rangle)\,(r'^2 \sin\theta'\, dr'\, d\phi'\, d\theta') \\
&= \frac{\rho}{4\pi\epsilon_0} \int_0^\pi \int_0^{2\pi} \int_0^R \frac{1}{\left[r'^2 \sin^2\theta' + (z - r'\cos\theta')^2\right]^{3/2}} \langle -r'\cos\phi'\sin\theta', -r'\sin\phi'\sin\theta', z - r'\cos\theta' \rangle\,(r'^2 \sin\theta'\, dr'\, d\phi'\, d\theta') \\
&= \frac{\rho}{4\pi\epsilon_0} \left\langle -\int_0^\pi \int_0^{2\pi} \int_0^R \frac{r'^3 \cos\phi' \sin^2\theta'}{\left[r'^2 \sin^2\theta' + (z - r'\cos\theta')^2\right]^{3/2}}\, dr'\, d\phi'\, d\theta', -\int_0^\pi \int_0^{2\pi} \int_0^R \frac{r'^3 \sin\phi' \sin^2\theta'}{\left[r'^2 \sin^2\theta' + (z - r'\cos\theta')^2\right]^{3/2}}\, dr'\, d\phi'\, d\theta', \right. \\
&\qquad \left. \int_0^\pi \int_0^{2\pi} \int_0^R \frac{r'^2 (z - r'\cos\theta')\sin\theta'}{\left[r'^2 \sin^2\theta' + (z - r'\cos\theta')^2\right]^{3/2}}\, dr'\, d\phi'\, d\theta' \right\rangle \\
&= \frac{\rho}{4\pi\epsilon_0} \left\langle -\left(\int_0^{2\pi} \cos\phi'\, d\phi'\right) \left\{ -\int_0^\pi \int_0^R \frac{r'^3 \sin^2\theta'}{\left[r'^2 \sin^2\theta' + (z - r'\cos\theta')^2\right]^{3/2}}\, dr'\, d\theta' \right\}, \right. \\
&\qquad -\left(\int_0^{2\pi} \sin\phi'\, d\phi'\right) \left\{ -\int_0^\pi \int_0^R \frac{r'^3 \sin^2\theta'}{\left[r'^2 \sin^2\theta' + (z - r'\cos\theta')^2\right]^{3/2}}\, dr'\, d\theta' \right\}, \\
&\qquad \left. \left(\int_0^{2\pi} d\phi'\right) \left\{ \int_0^\pi \int_0^R \frac{r'^2 (z - r'\cos\theta')\sin\theta'}{\left[r'^2 \sin^2\theta' + (z - r'\cos\theta')^2\right]^{3/2}}\, dr'\, d\theta' \right\} \right\rangle \\
&= \frac{\rho}{4\pi\epsilon_0} \left\langle -(0) \left\{ -\int_0^\pi \int_0^R \frac{r'^3 \sin^2\theta'}{\left[r'^2 \sin^2\theta' + (z - r'\cos\theta')^2\right]^{3/2}}\, dr'\, d\theta' \right\}, -(0) \left\{ -\int_0^\pi \int_0^R \frac{r'^3 \sin^2\theta'}{\left[r'^2 \sin^2\theta' + (z - r'\cos\theta')^2\right]^{3/2}}\, dr'\, d\theta' \right\}, \right. \\
&\qquad \left. (2\pi) \left\{ \int_0^\pi \int_0^R \frac{r'^2 (z - r'\cos\theta')\sin\theta'}{\left[r'^2 \sin^2\theta' + (z - r'\cos\theta')^2\right]^{3/2}}\, dr'\, d\theta' \right\} \right\rangle.
\end{aligned}$$

Continue the simplification.

$$\begin{aligned}
\mathbf{E} &= \frac{\rho}{4\pi\epsilon_0}\left\langle 0,0,2\pi\int_0^\pi\int_0^R \frac{r'^2(z-r'\cos\theta')\sin\theta'}{\left[r'^2\sin^2\theta'+(z-r'\cos\theta')^2\right]^{3/2}}\,dr'\,d\theta'\right\rangle \\
&= \frac{\rho}{2\epsilon_0}\langle 0,0,1\rangle\int_0^\pi\int_0^R \frac{r'^2(z-r'\cos\theta')\sin\theta'}{\left[r'^2\sin^2\theta'+(z-r'\cos\theta')^2\right]^{3/2}}\,dr'\,d\theta' \\
&= \frac{\rho\hat{\mathbf{z}}}{2\epsilon_0}\int_0^\pi\int_0^R \frac{r'^2(z-r'\cos\theta')\sin\theta'}{\left[r'^2\sin^2\theta'+(z-r'\cos\theta')^2\right]^{3/2}}\,dr'\,d\theta' \\
&= \frac{\rho\hat{\mathbf{z}}}{2\epsilon_0}\int_0^\pi\int_0^R \frac{r'^2(z-r'\cos\theta')\sin\theta'}{\left[r'^2(1-\cos^2\theta')+(z-r'\cos\theta')^2\right]^{3/2}}\,dr'\,d\theta' \\
&= \frac{\rho\hat{\mathbf{z}}}{2\epsilon_0}\int_0^R\left\{\int_0^\pi \frac{r'^2(z-r'\cos\theta')\sin\theta'}{\left[r'^2-r'^2\cos^2\theta'+(z-r'\cos\theta')^2\right]^{3/2}}\,d\theta'\right\}dr'
\end{aligned}$$

Make the following substitution.

$$\begin{aligned}
u &= z-r'\cos\theta' \quad\rightarrow\quad r'\cos\theta' = z-u \\
du &= r'\sin\theta'\,d\theta'
\end{aligned}$$

As a result,

$$\begin{aligned}
\mathbf{E} &= \frac{\rho\hat{\mathbf{z}}}{2\epsilon_0}\int_0^R\left\{\int_{z-r'\cos 0}^{z-r'\cos\pi}\frac{r'u}{\left[r'^2-(z-u)^2+u^2\right]^{3/2}}\,du\right\}dr' \\
&= \frac{\rho\hat{\mathbf{z}}}{2\epsilon_0}\int_0^R r'\left[\int_{z-r'}^{z+r'}\frac{u}{(2zu+r'^2-z^2)^{3/2}}\,du\right]dr'.
\end{aligned}$$

Make a second substitution.

$$\begin{aligned}
v &= 2zu+r'^2-z^2 &&\rightarrow &&\frac{v+z^2-r'^2}{2z} = u \\
dv &= 2z\,du &&\rightarrow &&\frac{dv}{2z} = du
\end{aligned}$$

So then

$$\begin{aligned}
\mathbf{E} &= \frac{\rho\hat{\mathbf{z}}}{2\epsilon_0}\int_0^R r'\left[\int_{2z(z-r')+r'^2-z^2}^{2z(z+r')+r'^2-z^2}\frac{v+z^2-r'^2}{2zv^{3/2}}\left(\frac{dv}{2z}\right)\right]dr' \\
&= \frac{\rho\hat{\mathbf{z}}}{8\epsilon_0 z^2}\int_0^R r'\left(\int_{z^2-2r'z+r'^2}^{z^2+2r'z+r'^2}\frac{v+z^2-r'^2}{v^{3/2}}\,dv\right)dr' \\
&= \frac{\rho\hat{\mathbf{z}}}{8\epsilon_0 z^2}\int_0^R r'\left\{\int_{z^2-2r'z+r'^2}^{z^2+2r'z+r'^2}\left[v^{-1/2}+(z^2-r'^2)v^{-3/2}\right]dv\right\}dr'.
\end{aligned}$$

Evaluate the integral and simplify the result.

$$\mathbf{E} = \frac{\rho\hat{\mathbf{z}}}{8\epsilon_0 z^2}\int_0^R r'\left[(2v^{1/2})\bigg|_{z^2-2r'z+r'^2}^{z^2+2r'z+r'^2} + (z^2 - r'^2)(-2v^{-1/2})\bigg|_{z^2-2r'z+r'^2}^{z^2+2r'z+r'^2}\right] dr'$$

$$= \frac{\rho\hat{\mathbf{z}}}{4\epsilon_0 z^2}\int_0^R r'\left[\left(\sqrt{z^2+2r'z+r'^2} - \sqrt{z^2-2r'z+r'^2}\right) - (z^2 - r'^2)\left(\frac{1}{\sqrt{z^2+2r'z+r'^2}} - \frac{1}{\sqrt{z^2-2r'z+r'^2}}\right)\right] dr'$$

$$= \frac{\rho\hat{\mathbf{z}}}{4\epsilon_0 z^2}\int_0^R r'\left\{\left[\sqrt{(z+r')^2} - \sqrt{(z-r')^2}\right] - (z^2 - r'^2)\left[\frac{1}{\sqrt{(z+r')^2}} - \frac{1}{\sqrt{(z-r')^2}}\right]\right\} dr'$$

$$= \frac{\rho\hat{\mathbf{z}}}{4\epsilon_0 z^2}\int_0^R r'\left\{\left[(z+r') - |z-r'|\right] - (z^2 - r'^2)\left(\frac{1}{z+r'} - \frac{1}{|z-r'|}\right)\right\} dr'$$

$$= \begin{cases} \frac{\rho\hat{\mathbf{z}}}{4\epsilon_0 z^2}\left\{\int_0^z r'\left\{\left[(z+r') - (z-r')\right] - (z^2 - r'^2)\left(\frac{1}{z+r'} - \frac{1}{z-r'}\right)\right\} dr' \right. \\ \qquad \left. + \int_z^R r'\left\{\left[(z+r') - (r'-z)\right] - (z^2 - r'^2)\left(\frac{1}{z+r'} - \frac{1}{r'-z}\right)\right\} dr'\right\} & \text{if } z < R \\ \frac{\rho\hat{\mathbf{z}}}{4\epsilon_0 z^2}\int_0^R r'\left\{\left[(z+r') - (z-r')\right] - (z^2 - r'^2)\left(\frac{1}{z+r'} - \frac{1}{z-r'}\right)\right\} dr' & \text{if } z > R \end{cases}$$

$$= \begin{cases} \frac{\rho\hat{\mathbf{z}}}{4\epsilon_0 z^2}\left\{\int_0^z r'\left[(2r') - (z^2 - r'^2)\left(\frac{-2r'}{z^2 - r'^2}\right)\right] dr' + \int_z^R r'\left[(2z) - (z^2 - r'^2)\left(\frac{-2z}{r'^2 - z^2}\right)\right] dr'\right\} & \text{if } z < R \\ \frac{\rho\hat{\mathbf{z}}}{4\epsilon_0 z^2}\int_0^R r'\left[(2r') - (z^2 - r'^2)\left(\frac{-2r'}{z^2 - r'^2}\right)\right] dr' & \text{if } z > R \end{cases}$$

$$= \begin{cases} \frac{\rho\hat{\mathbf{z}}}{4\epsilon_0 z^2}\left[\int_0^z r'(4r')\,dr' + \int_z^R r'(0)\,dr'\right] & \text{if } z < R \\ \frac{\rho\hat{\mathbf{z}}}{4\epsilon_0 z^2}\int_0^R r'(4r')\,dr' & \text{if } z > R \end{cases}$$

$$= \begin{cases} \frac{\rho\hat{\mathbf{z}}}{4\epsilon_0 z^2}\left(\frac{4z^3}{3}\right) & \text{if } z < R \\ \frac{\rho\hat{\mathbf{z}}}{4\epsilon_0 z^2}\left(\frac{4R^3}{3}\right) & \text{if } z > R \end{cases}$$

$$= \begin{cases} \left(\frac{q}{\frac{4}{3}\pi R^3}\right)\frac{\hat{\mathbf{z}}}{4\epsilon_0 z^2}\left(\frac{4z^3}{3}\right) & \text{if } z < R \\ \left(\frac{q}{\frac{4}{3}\pi R^3}\right)\frac{\hat{\mathbf{z}}}{4\epsilon_0 z^2}\left(\frac{4R^3}{3}\right) & \text{if } z > R \end{cases}$$

Therefore, the electric field at $\mathbf{r} = \langle 0, 0, z\rangle$ is

$$\mathbf{E} = \begin{cases} \dfrac{1}{4\pi\epsilon_0}\dfrac{q}{R^3}z\hat{\mathbf{z}} & \text{if } z < R \\ \\ \dfrac{1}{4\pi\epsilon_0}\dfrac{q}{z^2}\hat{\mathbf{z}} & \text{if } z > R \end{cases}.$$